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DUALITY UNDER DEPENDENCY INVERSION AND NOETHER THEORY FOR SECOND-ORDER LAGRANGIANS

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Abstract—We consider the duality which is obtained by interchanging dependent and independent variables in a Lagrangian. This duality is studied here for a second-order Lagrangian, which could describe, e.g. a Euler–Bernoulli beam (more generally, a one-dimensional elastic body of higher grade); however, the essential theory follows purely formally for a general Lagrangian. Next, the corresponding Noether theory of transformation groups and invariants is developed, making use of duality as far as possible, and showing the limits of duality theory. Particular emphasis is laid on the difference between relations equally valid on any curve and those valid only on the solution curve, stressing the difference between "true" and "apparent" invariants.

INTRODUCTION

The duality considered in this paper is the correspondence between two processes of continuum mechanics, in which the dependent variables (current position of material points) are interchanged with the independent variables (reference position of material points). We regard this duality as a (substantial) generalization of the "invariance under time-reversal" of classical field theory to continuum mechanics. The major difference between classical field theory (e.g. electrodynamics) and continuum mechanics (e.g. elasticity) is, in our opinion, this—that the latter is a two-field theory. For instance, whereas the electric field vector is defined purely in the tangential spaces ("the point of the arrow" has no meaning), the displacement field vector, while defined in tangential space, is projected into the embedding Euclidean space, where "the point of the arrow" is the current position of the material point which was at the "root of the arrow" in reference configuration. Essentially, present and reference configuration are each a field, connected by a vector of Euclidean space, and by such operators as the "shifters" (which are not necessary in a single-field theory). Thus under time reversal in elasticity it is not the same process run backwards that one should consider, but another, dual process: there is another body (with a different constitutive equation), which has as reference configuration (being its natural, i.e. undistorted and unstressed state) the current configuration of the original body, and deforms into the original configuration of the latter.

The duality considered here was found in 1967 by Shield[1]. Shield considered only hyperelastic bodies (whose inner energy is taken as Lagrange-function), and apparently did not yet realize the connection to the energy-momentum tensor, and thus to the invariant integrals of crack mechanics. This connection is fully and clearly demonstrated by Chadwick[2]. The extension to an arbitrary Lagrangian of first order was made by Shield[3] in 1977. We shall not enter into the details of Refs. [4–11]; essentially, the physical interpretation was developed, and the theory of forces and moments on the inhomogeneities was somewhat formalized. There still remains, however, a gap: are these forces the dual forces considered here? Is the dual body of this paper the inhomogeneity of the original body? At present, we do not know; we suspect that it is not the case, and that some conceptual confusions remain to be unraveled. Even if this were so, there should be some definite relations between the two correspondences.

The paper [10] was the starting point of this paper (see also the paper by Kienzler in this issue). We consider a one-dimensional case, so that the kinematics are irrelevant, and only the dynamics are of interest. The underlying idea was that of a beam: thus we took the displacement as dependent variable. This is at variance with duality, where the current position should be taken; accordingly, we (somewhat artificially) have to separately postulate that the relation displacement vs place be invertible. However, it turns out that the particular form of the Lagrange-function of a beam plays no rôle, so that we consider a general Lagrangian, although still keeping the notation of beam theory. The first section identifies the dual shear force with the Hamiltonian, i.e. the (one-component) energy-momentum tensor. Throughout the paper, we strictly differentiate between relations valid pointwise on extended configuration space, and relations that hold only on the solution curve.

The section on transformation groups and invariants follows[12]: essentially, we try to use duality as far as possible, and indeed some nice features do emerge. However, when it comes to giving meaningful examples, duality must be abandoned, at least insofar as the original body is considered the only one of interest.

DUALITY

Consider a family of Euclidean spaces $\mathbb{E}^{n+2}$ of dimension $n+2$, $n = 0, 1, 2, \ldots$, with coordinates $x, w, \overset{1}{w}, \overset{2}{w}, \ldots, \overset{n}{w}$. On each $\mathbb{E}^{n+2}$, consider the set of smooth functions into the reals, $\overset{n}{f}(x, w, \overset{1}{w}, \ldots, \overset{n}{w}) \in \mathbb{R}$. We introduce an operator

$$D^{(n)} := \partial/\partial x + \overset{1}{w}\,\partial/\partial w + \overset{2}{w}\,\partial/\partial \overset{1}{w} + \cdots + \overset{n+1}{w}\,\partial/\partial \overset{n}{w}$$

which acts on functions $\overset{n}{f}$ and produces particular functions $\overset{n+1}{f}$. For $n = 2$, we consider a particular function $L \in \{\overset{2}{f}\}$, called the "Lagrangian",

$$\overset{2}{L}: \{x, w, \overset{1}{w}, \overset{2}{w}\} \to \mathbb{R}.$$

We denote the partial derivatives by

$$q := -\partial \overset{2}{L}/\partial w, \qquad m := -\partial \overset{2}{L}/\partial \overset{1}{w}, \qquad M := \partial \overset{2}{L}/\partial \overset{2}{w};$$

these are likewise functions on $\mathbb{E}^4$. We define a function $E\langle \overset{2}{L} \rangle \in \{\overset{4}{f}\}$ by

$$E\langle \overset{2}{L} \rangle := -q + D^{(2)}m + D^{(3)}(D^{(2)}M)$$

and call it the "Eulerian" corresponding to $\overset{2}{L}$. One can arrive at it in two steps, by first defining

$$Q := -m - D^{(2)}M$$

(which maps $\mathbb{E}^5$ into $\mathbb{R}$); then

$$E\langle \overset{2}{L} \rangle = -q - D^{(3)}Q.$$

From now on, we shall omit the superscript 2 on L.

Now consider

$$D^{(2)}L = \partial L/\partial x - \overset{1}{w}q - \overset{2}{w}m + \overset{3}{w}M;$$

as is easily seen,

$$D^{(2)}L - \partial L/\partial x = \overset{1}{w}E\langle L \rangle + D^{(3)}(\overset{1}{w}Q + \overset{2}{w}M).$$

Let now another map from $\mathbb{E}^5$ into $\mathbb{R}$ be

$$\bar{Q} := L - \overset{1}{w}Q - \overset{2}{w}M;$$

then

$$\overset{1}{w}E\langle L\rangle = -\partial L/\partial x + D^{(3)}\bar{Q}.$$

Essentially, we want to study the equivalence

$$q + D^{(3)}Q = (\partial L/\partial x - D^{(3)}\bar{Q})/\overset{1}{w},$$

i.e. we take this as a pretext to introduce duality.

Consider a family of "dual" (in the sense specified below) Euclidean spaces $\bar{\mathbb{E}}^{n+2}$ of dimension $n+2$, $n = 0, 1, 2, \ldots$, with coordinates $\bar{w}, \bar{x}, \overset{1}{x}, \overset{2}{x}, \ldots, \overset{n}{x}$. On each $\bar{\mathbb{E}}^{n+2}$, let $\overset{\underline{n}}{\mathrm{f}}$ be an arbitrary smooth real-valued function, and let

$$\bar{D}^{(n)} := \partial/\partial\bar{w} + \overset{1}{x}\,\partial/\partial\bar{x} + \overset{2}{x}\,\partial/\partial\overset{1}{x} + \cdots + \overset{n+1}{x}\,\partial/\partial\overset{n}{x}$$

be the "dual" operator of $D^{(n)}$, with domain $\{\overset{\underline{n}}{\mathrm{f}}\}$ and range in $\{\overset{\underline{n\pm 1}}{\mathrm{f}}\}$. The "duality" we intend here is defined by the family of correspondences $\overset{n}{\Delta}: \bar{\mathbb{E}}^{n+2} \to \mathbb{E}^{n+2}$. In particular, for

$$n = 3: \qquad x = \bar{x}, \qquad w = \bar{w},$$

$$\overset{1}{w} = 1/\overset{1}{x}, \qquad \overset{2}{w} = -\overset{2}{x}/(\overset{1}{x})^3, \qquad \overset{3}{w} = 3(\overset{2}{x})^2/(\overset{1}{x})^5 - \overset{3}{w}/(\overset{1}{w})^4.$$

This correspondnce is reciprocal, i.e. its inverse $\overset{3}{\Delta}$ has the same form,

$$\bar{x} = x, \qquad \bar{w} = w,$$

$$\overset{1}{x} = 1/\overset{1}{w}, \qquad \overset{2}{x} = -\overset{2}{w}/(\overset{1}{w})^3, \qquad \overset{3}{x} = 3(\overset{2}{w})^2/(\overset{1}{w})^5 - \overset{3}{w}/(\overset{1}{w})^4.$$

For each $n = 0, 1, 2, \ldots$, $\overset{n+1}{\Delta}$ is a proper extension of $\overset{n}{\Delta}$, i.e. $\overset{n+1}{\Delta}$ consists of $\overset{n}{\Delta}$ and one additional relation between $\overset{n+1}{x}$ and $\overset{n+1}{w}$. We shall explain below how this $\overset{3}{\Delta}$ was arrived at, and how to extend the family; the algorithm is such that reciprocity is preserved. Apart from the first two relations, the gradient matrix is triangular, e.g. for $n = 2$:

$$\partial\overset{1}{x}/\partial\overset{1}{w} = -1/(\overset{1}{w})^2 = -(\overset{1}{x})^2;$$

$$\partial\overset{2}{x}/\partial\overset{1}{w} = 3\overset{2}{w}/(\overset{1}{w})^4 = -3\overset{1}{x}\overset{2}{x}, \qquad \partial\overset{2}{x}/\partial\overset{2}{w} = -1/(\overset{1}{w})^3 = -(\overset{1}{x})^3.$$

Again, this is reciprocal: the inverse matrix has the same form.

Again for $n = 2$, we prepare the correspondence between the operators $\bar{D}^{(2)}$, $D^{(2)}$:

$$\{D^{(2)}\langle\overset{\underline{2}}{\mathrm{f}}\cdot\overset{\underline{2}}{\Delta}\rangle\}\cdot\overset{2}{\Delta} = \{[\partial/\partial x + \overset{1}{w}\,\partial/\partial w + \overset{2}{w}\,\partial/\partial\overset{1}{w} + \overset{3}{w}\,\partial/\partial\overset{2}{w}]\overset{\underline{2}}{\mathrm{f}}[w, x, 1/\overset{1}{w}, -\overset{2}{w}/(\overset{1}{w})^3]\}\cdot\overset{2}{\Delta}$$

$$= \frac{\partial}{\partial\bar{x}}\overset{\underline{2}}{\mathrm{f}} + \frac{1}{\overset{1}{x}}\frac{\partial}{\partial\bar{w}}\overset{\underline{2}}{\mathrm{f}} - \frac{\overset{2}{x}}{(\overset{1}{x})^3}\left\{-(\overset{1}{x})^2\frac{\partial}{\partial\overset{1}{x}}\overset{\underline{2}}{\mathrm{f}} - 3\overset{1}{x}\overset{2}{x}\frac{\partial}{\partial\overset{2}{x}}\overset{\underline{2}}{\mathrm{f}}\right\}$$

$$+\left[\frac{3(\overset{2}{x})^2}{(\overset{1}{x})^5} - \frac{\overset{3}{x}}{(\overset{1}{x})^4}\right]\left\{-(\overset{1}{x})^3\frac{\partial}{\partial\overset{2}{x}}\overset{\underline{2}}{\mathrm{f}}\right\} = \frac{1}{\overset{1}{x}}\bar{D}^{(2)}\langle\overset{\underline{2}}{\mathrm{f}}\rangle.$$

In the same way, using the higher gradients $\partial\overset{3}{x}/\partial\overset{1}{w}$, etc., it can be shown that

$$\{D^{(3)}\langle\overset{3}{\mathrm{T}}\cdot\overset{3}{\bar{\Delta}}\rangle\}\cdot\overset{3}{\Delta} = (1/\overset{1}{x})\bar{D}^{(3)}\langle\overset{3}{\mathrm{T}}\rangle;$$

note that these correspondences are reciprocal.

Now we define the dual Lagrangian function by

$$\bar{L} := \overset{1}{x}L\cdot\overset{2}{\Delta},$$

i.e.

$$\bar{L}(\bar{w},\bar{x},\overset{1}{x},\overset{2}{x}) = \overset{1}{x}L(\bar{x},\bar{w},1/\overset{1}{x},-\overset{2}{x}/(\overset{1}{x})^2).$$

This relation is again reciprocal, i.e. the dual of $\bar{L}$ is L.

In the following, we shall not show the correspondence maps $\overset{n}{\Delta}$ or $\overset{n}{\bar{\Delta}}$ explicitly anymore; it is intended that either the same or corresponding independent variables should appear in any equation. Thus we shall write, e.g.

$$\bar{D}^{(3)}\langle\overset{3}{\mathrm{T}}\rangle = (1/\overset{1}{w})D^{(3)}\langle\overset{3}{\mathrm{T}}\rangle.$$

Starting from $\bar{L}$ we define the dual quantities

$$\begin{aligned}
\bar{q} &:= -\partial\bar{L}/\partial\bar{x} = -(1/\overset{1}{w})\,\partial L/\partial x,\\
\bar{m} &:= -\partial\bar{L}/\partial\overset{1}{x} = -\overset{1}{x}(\partial L/\partial\overset{1}{w})[-1/(\overset{1}{x})^2]-\overset{1}{x}(\partial L/\partial\overset{2}{w})[3\overset{2}{x}/(\overset{1}{x})^4]-L\\
&\quad = 3\overset{2}{w}M-\overset{1}{w}m-L,\\
\bar{M} &:= \partial\bar{L}/\partial\overset{2}{x} = \overset{1}{x}(\partial L/\partial\overset{2}{w})[-1/(\overset{1}{x})^3] = -(\overset{1}{w})^2M.
\end{aligned}$$

(The relations are reciprocal.) Furthermore, we introduce the dual Eulerian

$$E\langle\bar{L}\rangle := -\bar{q}+D^{(2)}\bar{m}+\bar{D}^{(3)}(\bar{D}^{(2)}\bar{M}).$$

Eventually, we should like to define

$$\bar{Q} := \bar{m}-\bar{D}^{(2)}\bar{M},$$

so that

$$E\langle\bar{L}\rangle = -\bar{q}-\bar{D}^{(3)}\bar{Q};$$

but $\bar{Q}$ was already defined above. However, the two definitions are equivalent:

$$\begin{aligned}
\bar{Q} = L+\overset{1}{w}(m+D^{(2)}M)-\overset{2}{w}M &= \bar{L}/\overset{1}{x}+(1/\overset{1}{x})\{-\bar{L}-\overset{1}{x}\bar{m}+3\overset{2}{x}\bar{M}\\
&\quad +(1/\overset{1}{x})\bar{D}^{(2)}[-(\overset{1}{x})^2\bar{M}]\}+[\overset{2}{x}/(\overset{1}{x})^3][-(\overset{1}{x})^2\bar{M}] = -\bar{m}-\bar{D}^{(2)}\bar{M}.
\end{aligned}$$

Thus we may now write the equivalence we started from in the self-dual form

$$q+D^{(3)}Q = -\bar{q}-\bar{D}^{(3)}\bar{Q},$$

or

$$E\langle L\rangle = -E\langle\bar{L}\rangle.$$

This concludes our introduction of duality.

What is the meaning of this duality?

To begin with, the fundamental correspondence $\overset{n}{\Delta}$ is obtained as follows. Consider an arbitrary, sufficiently smooth curve $w = \omega(x)$, on a segment $0 \leqslant x \leqslant l$, say; and suppose ω is uniquely invertible on its range, with correspondingly smooth inverse $x = \varkappa(w)$. Then, if $\overset{n}{w}$ is identified with $\mathrm{d}^n\omega/\mathrm{d}x^n$, it turns out that also $\overset{n}{x}$ is identified with $\mathrm{d}^n\varkappa/\mathrm{d}w^n$, i.e. $\overset{n}{\Delta}$ was set up in this way. Thus this duality results from interchanging dependent and independent variable in the (arbitrary) curve. However, it was our intention to show that all essential relations are identities on the underlying space $\mathbb{E}^n$ or $\bar{\mathbb{E}}^n$, or, that they are equalities between maps (not between values), completely independent from, but compatible with any underlying curve. The significance of this duality stems from this, that $\bar{Q}$ turns out to be the Hamiltonian of L (and likewise Q of $\bar{L}$), or, the one component of the energy-momentum tensor (see Ref. [12], p. 120, eqn (6.19); and Ref. [14], p. 326, eqn (3.6)). Suppose Q has some physical meaning w.r.t. L: e.g. if L is the Lagrangian of a Euler beam, then Q is the distributed shear force. What we have shown, then, is this, that there is another Lagrangian $\bar{L}$ such that the Hamiltonian $\bar{Q}$ of L has w.r.t. $\bar{L}$ the same meaning as Q had to L. By the same token, q is the external distributed load on the beam, its dual $\bar{q}$ is, essentially, the inhomogeneity distribution (place-dependence of stiffness, external load and moment) of the original beam. However, as the following examples show, any particular form of L need not be preserved by duality; in particular not the Lagrangian of a beam. On the other hand, this "shortcoming" is worsened here by the fact that, for the sake of simplicity, we took what for the beam would be the deflection as variable w, instead of taking the new position, as we should have done. It appears that duality can be properly established only in two (plane strain) or three dimensions. However, this work should at least intuitively show the way.

PARTICULAR LAGRANGIANS; THE GAUGE FUNCTION

To fix ideas, we look at some special classes of Lagrangians. We always assume that L is regular, i.e. the "Hessian" $\partial^2 L/(\partial \overset{2}{w})^2 \neq 0$, $0 < x < l$; that is, $\partial M/\partial \overset{2}{w} \neq 0$. We begin with the requirement that M depend only on x and $\overset{2}{w}$, not on w and $\overset{1}{w}$. Then q and m do not depend on $\overset{2}{w}$, and we are led to the "separable"

$$L_s = W(x, \overset{2}{w}) - V(x, w, \overset{1}{w}).$$

Now we separately look for special forms of W and V. Major simplifications occur if W is homogeneous, of order h, say in $\overset{2}{w}$:

$$W_h = (1/h)s(x)(\overset{2}{w})^h, \qquad L_h = W_h - V; \qquad M\overset{2}{w} = W_h/h,$$

with arbitrary $s(x) \neq 0$ on $0 \leqslant x \leqslant l$. Obviously, $h \neq 0$, and $h \neq 1$ for regularity. On the other hand, physical interpretation suggests that m, q be "controllable", i.e. explicit (known) functions of x only. Then

$$V_c = m(x)\overset{1}{w} + q(x)w, \qquad L_c = W - V_c.$$

This suffices to explicitly integrate the "kinetic" equations,

$$\hat{Q}(x) = \int_x^l q(\xi)\,\mathrm{d}\xi + Q_l,$$

$$\hat{M}(x) = \int_x^l [(\xi - x)q(\xi) + m(\xi)]\,\mathrm{d}\xi + M_l + (l - x)Q_l.$$

Thanks to regularity, at least in principle we can solve

$$M(x, \overset{2}{w}) = \partial W(x, \overset{2}{w})/\partial \overset{2}{w} = \hat{M}(x)$$

for w, obtaining the normal form $\overset{2}{w}$ = function of x. Thus, if now $\overset{2}{w}$ is identified with $d^2\omega/dx^2$, this special "kinematic" equation is solved by two quadratures. Eventually, combining these two cases, let

$$L_{hc} = W_h - V_c = (1/h)s(x)(\overset{2}{w})^h - m(x)\overset{1}{w} - q(x)w.$$

Obviously, the simplest case is $h = 2$, when $M(x, \overset{2}{w}) = s(x)\overset{2}{w}$, so that the normal form is immediate (and one-valued). This is the Lagrangian for the so-called Euler–Bernoulli beam, w being the deflection, $s(x)$ the stiffness and $q(x)$ the external distributed load; the external distributed moment $m(x)$ is usually absent in technical applications. Since we are not interested in solving the kinematic equation here, we may stay with L_{hc}.

Now let us look at the dual.

$$\bar{L}_{hc} := (1/h)s(\bar{x})(-\overset{2}{x})^h/(\overset{1}{x})^{3h-1} - m(\bar{x}) - \overset{1}{x}q(\bar{x})\bar{w}$$

certainly looks awkward—it is not even separable. It becomes "dually homogeneous/controllable" only if $h = 1/3$, so that only then duality will preserve particular results essentially due to the special form of L_{hc}. In general, with $(\,)' \equiv d(\,)/dx$,

$$\bar{q} = (1/h)s'(\bar{x})(-\overset{2}{x})^h/(\overset{1}{x})^{3h-1} - m'(\bar{x}) - q'(\bar{x})\bar{w}\overset{1}{x},$$

$$\bar{m} = -(3 - 1/h)s(\bar{x})(-\overset{2}{x})^h/(\overset{1}{x})^{3h} - q(\bar{x})\bar{w}.$$

q is not controllable, neither "dually" nor directly; but we may say it is "observable"— it can easily be seen if L depends on x or not; thus, while q may not be prescribed in general, it is possible to require q to vanish identically, i.e. to accept only such Lagrangians that are independent of x.

So far, we considered the Lagrangian as primarily given, whence the Eulerian was derived. Let us now consider the explicit form of the Eulerian as primary, and look for the Lagrangians that lead to it. As is easily checked, Lagrangians having the same Eulerian differ at most by a (or: are determined up to an additive) total derivative of a so-called gauge function $g(x, w, \overset{1}{w})$, being an arbitrary (sufficiently smooth) function of its three arguments, essentially independent of $\overset{2}{w}$:

$$E\langle L_{(1)}\rangle \equiv E\langle L_{(2)}\rangle \leftrightarrow L_{(1)} - L_{(2)} = D^{(1)}g, \qquad g = g(x, w, \overset{1}{w}).$$

Then to q, m, M are added the terms

$$\tilde{q} := -\partial(D^{(1)}g)/\partial w = -D^{(1)}(\partial g/\partial w),$$

$$\tilde{M} := \partial(D^{(1)}g)/\partial \overset{2}{w} = \partial g/\partial \overset{1}{w},$$

$$\tilde{m} := -\partial(D^{(1)}g)/\partial \overset{1}{w} = -D^{(1)}(\partial g/\partial \overset{1}{w}) - \partial g/\partial w = -D^{(1)}\tilde{M} - \partial g/\partial w,$$

and thus

$$\tilde{q} = D^{(1)}(\tilde{m} + D^{(1)}\tilde{M}), \qquad \text{or:} \quad E\langle D^{(1)}g\rangle \equiv 0.$$

The structure of L_s is preserved by gauging if $g(x, w, \overset{1}{w}) = g_1(\overset{1}{w}) + g_2(x, w)$; for L_h we require $g_1 \equiv 0$ since we exclude $h = 1$; *for* L_c, $g_2(x, w) = g_3(x)w + g_4(x)$; thus eventually the form of L_{hc} is preserved if g does not depend on $\overset{1}{w}$ and is linear in w ("restricted", or "kinetic" gauge group).

If we "dualize" g just as we did with L, we find that, since dualization commutes with differentiation w.r.t. x or w, dualizing and gauging are compatible, or: the dual of a gauge function is a gauge function of the dual. However, this does not help to bring the dual Lagrangian found above to a nicer shape. We only remark that q can be made to vanish on any fixed curve by adding a suitable function of x to the Lagrangian (if, as here assumed, it is at all permissible to change the Lagrangian in the problem at hand).

INFINITESIMAL TRANSFORMATION GROUPS AND INVARIANTS

In this section it is convenient to introduce curves, as already hinted at in the concluding remarks in the first section. Consider an arbitrary real valued, sufficiently smooth function ω defined on the real segment $\mathbb{B} = \{x \mid 0 \leqslant x \leqslant l\}$, and its extension

$$\omega_0^{\{n\}} : \mathbb{B} \to \mathbb{E}^{n+1}, \qquad x \mapsto \{x, \omega(x), \mathrm{d}\omega/\mathrm{d}x, \ldots, \mathrm{d}^n\omega/\mathrm{d}x^n\}.$$

We shall write $\hat{=}$ for relations which are meaningful only if a curve, any one, is laid down, and $()' \equiv \mathrm{d}()/\mathrm{d}x$ for a function of x only. For instance, for any function $\overset{n}{\mathrm{f}}$ defined on $\mathbb{E}^{n+2}$,

$$\mathrm{d}(\overset{n}{\mathrm{f}} \cdot \omega_0^{\{n\}})/\mathrm{d}x \doteq (D^{(n)}\overset{n}{\mathrm{f}}) \cdot \omega_0^{\{n+1\}} \hat{=} \overset{n}{\mathrm{f}}'.$$

We introduce two notation schemes for transformations, the "flow" and the "variation" notation, since both will be convenient. In "flow" notation, we consider the infinitesimal transformation group of degree p with parameter ε

$$\hat{x} = x + \varepsilon\tau(x, w, \overset{1}{w}, \ldots, \overset{p}{w}),$$

$$\hat{w} = w + \varepsilon\xi(x, w, \overset{1}{w}, \ldots, \overset{p}{w}),$$

(usually, $p = 0$). In "variation" notation, ξ is the "total" variation of a curve $\omega(\cdot)$, which is composed of a "local" (or "syntopic", or "synchronous") variation and a "convective" (or 'asyntopic", or "asynchronous") variation,

$$\xi \hat{=} \delta\langle\omega\rangle = \delta^l\langle\omega\rangle + \delta^c\langle\omega\rangle.$$

δ^l effects an actual change of the function $\omega(\cdot)$, the new function, however, being evaluated at the same point x; δ^c describes the change of the value of the same function ω when evaluated at $\hat{x}$, and attributes this value to x (in yet another terminology, $\delta^c\langle\omega\rangle$ is the "pullback of $\omega(\cdot)$ under the flow τ"). Each is characterized by a particular property. One usually assumes that the local variation commutes with differentiation,

$$(\delta^l\langle\omega\rangle)' \hat{=} \delta^l\langle\omega'\rangle, \qquad \equiv \delta^l\langle\omega\rangle',$$

i.e. one considers only local variations that satisfy this requirement. (This is the same as saying: we define $\delta^l\langle\omega'\rangle$ by the above relation; or: we choose to identify the (*a priori*) abstract symbol $\delta^l\langle\omega'\rangle$ with the "lifting" $(\delta^l\langle\omega\rangle)'$, just as we did when identifying $\overset{1}{w}$ with ω'.) This should hold recursively for higher derivatives as well. The convective variation is characterized (as a pullback operator) by its vectorfield; formally,

$$(\partial\delta^c/\partial\varepsilon)|_{\varepsilon=0} \hat{=} \tau()';$$

hence we obtain, for instance, the formula

$$\delta^c\langle\omega\rangle \hat{=} \tau\omega'$$

as is well known in continuum mechanics for the convected derivative. This again holds for higher derivatives as well. Thus we get

$$\delta^l\langle\omega\rangle \mathrel{\hat{=}} \xi-\tau\omega' ;$$

differentiating and identifying $\tau\omega''$ with $\delta^c\langle\omega'\rangle$,

$$\delta\langle\omega'\rangle \mathrel{\hat{=}} \xi'-\tau'\omega' ;$$

similarly,

$$\delta\langle\omega''\rangle \mathrel{\hat{=}} \xi''-2\tau'\omega''-\tau''\omega'.$$

After this intuitive procedure, we want to repeat these steps without a curve. We only need to lay down the axioms

0. $\delta\langle w\rangle = \xi(x, w, \overset{1}{w}, \ldots, \overset{p}{w});$
1. $\delta\langle\cdot\rangle = \delta^l\langle\cdot\rangle+\delta^c\langle\cdot\rangle;$
2. $\delta^l\langle\overset{\nu+1}{w}\rangle = D^{(n+\nu)}\delta^l\langle\overset{\nu}{w}\rangle, \qquad \nu = 0, 1, 2, \ldots;$
3. $\delta^c\langle\overset{\nu}{w}\rangle = \overset{\nu+1}{w}\tau(x, w, \overset{1}{w}, \ldots, \overset{p}{w}), \qquad \nu = 0, 1, 2, \ldots$

to find

$$\delta^l\langle w\rangle = \xi-\overset{1}{w}\tau,$$
$$\delta\langle\overset{1}{w}\rangle = D^{(p)}\xi-\overset{1}{w}D^{(p)}\tau,$$
$$\delta\langle\overset{2}{w}\rangle = D^{(p+1)}D^{(p)}\xi-2\overset{2}{w}D^{(p)}\tau-\overset{1}{w}D^{(p+1)}D^{(p)}\tau.$$

Finally, coming to a more physical ground, let us consider the "action" integral with boundary terms

$$A \mathrel{\hat{=}} \int_0^l L\cdot\omega_0^{\{2\}}\,\mathrm{d}x-(Q_*\omega+M_*\omega')|_0^l,$$

where Q_*, M_* are external variables, defined only on the end points; we shall see how much freedom we have in choosing them.

The variation of A, which we denote by $\tilde{\mathscr{A}}_\varepsilon\langle A\rangle$, is

$$\tilde{\mathscr{A}}_\varepsilon\langle A\rangle := \int_{\varepsilon\tau[0,\,\omega(0)]}^{l+\varepsilon\tau[l,\omega(l)]} L[\hat{x}, \hat{\omega}(\hat{x}), \ldots]\,\mathrm{d}\hat{x}-\int_0^l L[x, \omega(x), \ldots]\,\mathrm{d}x-\varepsilon[Q_*\delta^l\langle\omega\rangle+M_*\delta^l\langle\omega\rangle']|_0^l,$$
$$\equiv \int_0^l \{L(\hat{x}, \hat{\omega}(\hat{x}), \ldots)|_{\hat{x}=\hat{x}(x)}\,\mathrm{d}\hat{x}/\mathrm{d}x-L[x, \omega(x), \ldots]\}\,\mathrm{d}x-\varepsilon(Q_*\delta^l\langle\omega\rangle+M_*\delta^l\langle\omega\rangle')|_0^l.$$

Let us denote the integrand in the second expression by $\mathscr{A}_\varepsilon\langle L\rangle$, where $\mathscr{A}_\varepsilon$ is an operator which depends on the group parameter ε. Obviously, for $\varepsilon = 0$, $\mathscr{A}_\varepsilon$ is the zero-operator; thus we are led to consider the derived operator

$$\mathscr{D} := (\mathrm{d}\mathscr{A}_\varepsilon/\mathrm{d}\varepsilon)|_{\varepsilon=0}.$$

The result of applying $\mathscr{D}$ on L can be evaluated in two ways: either by

$$\begin{aligned}\mathscr{D}\langle L\rangle \mathrel{\hat{=}} &(\mathrm{d}/\mathrm{d}\varepsilon)\{[L+\varepsilon(\tau D^{(2)}L+\delta'\langle w\rangle\,\partial L/\partial w+\delta'\langle \overset{1}{w}\rangle\,\partial L/\partial \overset{1}{w}\\ &+\delta'\langle \overset{2}{w}\rangle\,\partial L/\partial \overset{2}{w})+o(\varepsilon)](1+D^{(p)}\tau)-L\}|_{\varepsilon=0}\cdot\omega_0^{\{\nu+2\}},\\ &=\{D^{(p)}\tau L+\tau D^{(2)}L-\delta'\langle w\rangle q-\delta'\langle \overset{1}{w}\rangle m+\delta'\langle \overset{2}{w}\rangle M\}\cdot\omega_0^{\{\nu+2\}},\end{aligned}$$

or by

$$\begin{aligned}\mathscr{D}\langle L\rangle = &\,\mathrm{d}/\mathrm{d}\varepsilon\{[L+\varepsilon(\tau\,\partial L/\partial x+\delta\langle w\rangle\,\partial L/\partial w+\delta\langle \overset{1}{w}\rangle\,\partial L/\partial \overset{1}{w}\\ &+\delta\langle \overset{2}{w}\rangle\,\partial L/\partial \overset{2}{w})+o(\varepsilon)](1+\varepsilon D^{(p)}\tau)-L\}|_{\varepsilon=0}\cdot\omega_0^{\{\nu+2\}},\\ &=\{D^{(p)}\tau L+\tau\,\partial L/\partial x-\delta\langle w\rangle q-\delta\langle \overset{1}{w}\rangle m+\delta\langle \overset{2}{w}\rangle M\}\cdot\omega_0^{\{\nu+2\}},\end{aligned}$$

where $\nu = \max(p, 2)$. (It is a trivial calculation to show that the two expressions give the same result; we shall do this at the end of this section.)

To begin with, we consider the first one. By straightforward evaluation one establishes, for an arbitrary function $\mathrm{f}(x, w, \overset{1}{w}, \ldots, \overset{m}{w})$, $0 \leqslant m \leqslant p$, the identity on $\mathbb{E}^{\nu+4}$

$$-q\mathrm{f}-mD^{(\nu)}\mathrm{f}+MD^{(\nu+1)}D^{(\nu)}\mathrm{f} = E\langle L\rangle\mathrm{f}+D^{(\nu+1)}(Q\mathrm{f}+MD^{(\nu)}\mathrm{f}).$$

Now let $\mathrm{f} = \delta'\langle\omega\rangle$; recalling that δ' commutes with differentiation,

$$\mathscr{D}\langle L\rangle \mathrel{\hat{=}} \{E\langle L\rangle\delta'\langle w\rangle+D^{(\nu+1)}(L\tau+Q\delta'\langle w\rangle+M\delta'\langle \overset{1}{w}\rangle)\}\cdot\omega_0^{\{\nu+2\}}.$$

We call the contents of the round brackets the "Noetherian", and denote it by N, defined on $\mathbb{E}^{\nu+3}$,

$$N := L\tau+Q\delta'\langle w\rangle+M\delta'\langle \overset{1}{w}\rangle, \qquad \equiv N\langle L, \tau, \xi, \overset{\nu+1}{w}\rangle:$$

N is a functional of L, τ and ξ, and depends explicitly (in fact, linearly) on $\overset{\nu+1}{w}$. Returning to the variation of the action A, we may define also for $\tilde{\mathscr{A}}_\varepsilon$ the derived operator

$$\tilde{\mathscr{D}} := (\mathrm{d}\tilde{\mathscr{A}}_\varepsilon/\mathrm{d}\varepsilon)|_{\varepsilon=0};$$

then, integrating N' on ω,

$$\tilde{\mathscr{D}}\langle L\rangle \mathrel{\hat{=}} \int_0^l \{E\langle L\rangle\delta'\langle w\rangle\}\cdot\omega_0^{\{\nu+2\}}\,\mathrm{d}x+\{L\tau+(Q-Q_*)\delta'\langle w\rangle+(M-M_*)\delta'\langle \overset{1}{w}\rangle\}\cdot\omega_0^{\{\nu+1\}}|_0^l.$$

The principle of virtual work requires $\tilde{\mathscr{D}}\langle A\rangle = 0$ for arbitrary local variations $\delta'\langle\omega\rangle$ in the interior of $(0, l)$, provided τ vanishes identically in $w, \overset{1}{w}, \ldots, \overset{p}{w}$ at the endpoints. With the so-called fundamental lemma there follows Euler's equation and the "rigid" and/or "natural" boundary conditions, as is well known. We stress that $\tilde{\mathscr{D}}\langle A\rangle = 0$ does not imply $\mathscr{D}\langle L\rangle = 0$; on the contrary, N must match the boundary conditions, which generally implies that $N' \neq 0$.

Whereas this principle thus fixes a particular curve, the "orbit", while leaving the variations τ, ξ (almost) arbitrary, we now turn to the "opposite" problem, to somehow fix τ, ξ while leaving the curve arbitrary. One might require $\tilde{\mathscr{D}}\langle A\rangle = 0$ provided the external end terms Q_*, M_* vanish at both ends; here we choose to directly require that $\mathscr{D}\langle L\rangle = 0$ on $\mathbb{E}^{\nu+4}$, and to interpret this as a condition which the transformation group generators τ, ξ and the Lagrangian L have to satisfy. Thus, here we use the second of the two expressions for $\mathscr{D}\langle L\rangle$ derived above; denoting it with K (for "Killing"),

$$K := LD^{(p)}\tau - \tau\overset{1}{w}\bar{q} - q\delta\langle w\rangle - m\delta\langle\overset{1}{w}\rangle + M\delta\langle\overset{2}{w}\rangle, \qquad \equiv K\langle L, \tau, \xi; \overset{p+1}{w}, \overset{p+2}{w}\rangle.$$

K depends explicitly on $\overset{p+1}{w}$ and $\overset{p+2}{w}$ if and only if $p \geqslant 2$, it depends explicitly only on $\overset{3}{w}$ if $p = 1$. Whenever $\overset{p+2}{w}$ occurs explicitly, the dependence on it is linear. Anyway, K is defined in $\mathbb{E}^{p+4}$, i.e. $K = K(x, w, \overset{1}{w}, \ldots, \overset{p+2}{w})$. On the other hand, in $E\langle L\rangle\delta^l\langle w\rangle$ the highest superscript of w is max $(p, 4)$, whereas in $D^{(\nu+1)}N$ it is $\nu+2$; the w with the highest superscript always occurs explicitly and, in fact, linearly. We mention these details because one might doubt whether the equality on an arbitrary curve of the two expressions for L carries over into an identity, valid pointwise in a sufficiently high-dimensional space,

$$K \equiv E\langle L\rangle\delta^l\langle w\rangle + D^{(\nu+1)}N.$$

Considering only the coefficients of the leading w (with the highest superscript), for $p \leqslant 1$ they appear in $D^{(\nu+1)}N$ and in $E\langle L\rangle\delta^l\langle w\rangle$, being $\mp(\partial M/\partial\overset{2}{w})\delta^l\langle w\rangle$ (which multiplies $\overset{4}{w}$) and cancelling each other; for $p \geqslant 3$ they appear in $D^{(\nu+1)}N$ and in K, being $M(\partial\xi/\partial\overset{p}{w} - \overset{1}{w}\,\partial\tau/\partial\overset{p}{w})$ (which multiplies $\overset{p+2}{w}$) and again cancelling each other; for $p = 2$ several terms are involved. We shall not analyze this further, since we shall prove the general identity below: here we merely wanted to show that this generalization is not trivial.

Now let us return to our original point: we wanted to require $K = 0$ on $\mathbb{E}^{p+4}$ as a condition on L, τ and ξ. Whereas this yields only one equation if $p = 0$, it gives two equations (one extra for the coefficient of $\overset{3}{w}$) if $p = 1$; for $p \geqslant 2$, the situation becomes complicated, since $\overset{p+1}{w}$ need not occur linearly. This interpretation of $K = 0$ as a point identity is the one called "strict" in Ref. [13]. There, only the case $p = 1$ is considered, and $K \equiv 0$ leads to the system (63), (64); however, the Lagrangian is of first order. The authors state that in general, $p > 1$ "would not substantially enlarge the picture", while if one required "higher-order tangency of curves" to be preserved, this would be incompatible with $p > 0$ if one has more than one degree of freedom. We do not know how this theory should be expanded to our case; however, it may be that going to more degrees of freedom would essentially change some aspects. A minor difference between this paper and Ref. [13] is that we have tacitly suppressed the gauge function.

If now L, τ and ξ "match" so that $K \equiv 0$, then

$$D^{(\nu+1)}N + E\langle L\rangle\delta^l\langle w\rangle \equiv 0.$$

We call this the "evolution equation" for N. Since it is a pointwise identity on $\mathbb{E}^{\nu+4}$, it is valid for any curve (or, it is an identity also on the function space of curves). In particular, let us consider those special curves $\mathring{\omega}$ for which

$$E\langle L\rangle\cdot\mathring{\omega}_0^{\{2\}} \doteq 0:$$

these are the solution curves, or "orbits" ($\doteq$ shall denote equality on orbits only). On the orbits the evolution equation reduces to a conservation equation,

$$(D^{(\nu+1)}N)\cdot\mathring{\omega}_0^{\{\nu+2\}} \doteq N' \doteq \text{const}.$$

The constant value still must match the external end quantities Q_*, M_*, which thus cannot be independent. In general, they must satisfy some compatibility conditions (in mechanics, these are the global equilibrium conditions). Thus we say that N is conserved, or that it is invariant, only if it satisfies (identically) the evolution equation (and the boundary conditions are compatible). The essential point is that $\tilde{\mathscr{D}}\langle A\rangle = 0$ does not imply $\mathscr{D}\langle L\rangle = 0$ (as remarked above), so the latter condition must be imposed independently of the former. The obvious difference is that while $\tilde{\mathscr{D}}\langle A\rangle = 0$ determines the orbits, the evolution equation can be written explicitly without knowing the orbits at all. To clarify this point further, let us suppose $K = 0$ does not hold: Lagrangian and transformation group do not "match properly". Then we have

$$K \equiv E\langle L\rangle\delta'\langle w\rangle + D^{(\nu+1)}N,$$

valid not only for any curve, but also for any τ, ξ. In particular, on the orbits

$$N' \doteq K.$$

This we call the "equation of balance" for N; it is defined only on the orbit, but valid for arbitrary τ, ξ. Keeping to our definition, we should regard the case that K happens to vanish on the orbits, though not elsewhere, as a mere exception, and not call N invariant unless indeed $K \equiv 0$. In fact, the equation of balance is not any condition at all: it states the identity of the two expressions for $\mathscr{D}\langle L\rangle$ (which we shall prove in general just below) in the particular case when one considers only orbits; no further restriction is imposed. The equation of balance is thus trivial, whereas the evolution equation is not. The only significance of the equation of balance is this, that it gives us an easily computable expression for the rate of change of the Noetherian along an orbit; N may well have (for some particular τ, ξ) a meaning for physical interpretation.

Eventually, we introduce duality. Inserting $\delta'\langle w\rangle = \xi - \tau\overset{1}{w}$ in N we obtain

$$\begin{aligned} N &= L\tau + Q(\xi - \tau\overset{1}{w}) + M(D^{(\wp)}\xi - \overset{1}{w}D^{(\wp)}\tau - \overset{2}{w}\tau) \\ &= \bar{Q}\tau + Q\xi + M(D^{(\wp)}\xi - \overset{1}{w}D^{(\wp)}\tau), \\ &= Q\xi + MD^{(\wp)}\xi + \bar{Q}\tau + \bar{M}\bar{D}^{(\wp)}\tau, \end{aligned}$$

recalling that $MD^{(\wp)}w + \bar{M}\bar{D}^{(\wp)}x = 0$ and $\overset{1}{x}D^{(\wp)}\tau = \bar{D}^{(\wp)}\tau$. Thus the Noetherian is self-dual, if by duality one also interchanges τ and ξ. In the same way, using

$$\bar{D}^{(\wp+1)}\bar{D}^{(\wp)}\tau = (D^{(\wp+1)}D^{(\wp)}\tau)/(\overset{1}{w})^2 - (D^{(\wp)}\tau)\overset{2}{w}/(\overset{1}{w})^3,$$

we find

$$K = MD^{(\wp+1)}D^{(\wp)}\xi - mD^{(\wp)}\xi - q\xi + \overset{1}{w}(\bar{M}\bar{D}^{(\wp+1)}\bar{D}^{(\wp)}\tau - \bar{m}\bar{D}^{(\wp)}\tau - \bar{q}\tau).$$

Now it is sufficient to consider

$$\begin{aligned} D^{(\nu+1)}(Q\xi + MD^{(\wp)}\xi) &= MD^{(\wp+1)}D^{(\wp)}\xi + QD^{(\wp)}\xi + (D^{(3)}Q + q)\xi - q\xi \\ &\quad + (D^{(2)}M + Q + m)D^{(\wp)}\xi - (Q+m)D^{(\wp)}\xi \\ &= MD^{(\wp+1)}D^{(\wp)}\xi - mD^{(\wp)}\xi - q\xi - E\langle L\rangle\xi \end{aligned}$$

and the dual of this equation to show that the two expressions for $\mathscr{D}\langle L\rangle$ derived at the beginning are identical, and thereby to show that the equation of balance holds trivially.

THE INHOMOGENEOUS LINEAR TRANSFORMATION GROUP

As an example, we choose to study the group

$$\begin{aligned} \tau &= \bar{c} + \bar{g}w + \bar{k}x, \\ \xi &= c + gx + kw, \end{aligned}$$

$\bar{c}, \ldots, k$ constants. On the other hand, we start with an arbitrary Lagrangian, and shall then restrict it so as to satisfy Killing's equation $K = 0$. (Usually, the Lagrangian is given, and one has to find the group τ, ξ.) We find

$$N = cQ+g(Qx+M)+k(Qw+M\overset{1}{w})+\text{dual terms},$$

$$K = -cq-g(qx+m)+k(M\overset{2}{w}-m\overset{1}{w}-qw)+\overset{1}{w}(\text{dual terms}).$$

It is now a simple matter to satisfy $K \equiv 0$. For instance, consider only the c and g terms: q and m are at least easily "observable" (one sees immediately whether L depends on w, $\overset{1}{w}$ or not); if

$$L = L_s = W-V \,\&\, V = 0: \qquad L = W(x, \overset{2}{w}),$$

with $\xi = c+gx$, $\tau = 0$ we find the evolution equation

$$D^{(3)}[cQ+g(Qx+M)]+(c+gx)D^{(3)}D^{(2)}(\partial W/\partial \overset{2}{w}) \equiv 0.$$

Hence on the orbit, taking c and g independently,

$$Q \doteq \text{const.}, \qquad Qx+M \doteq \text{const.},$$

provided $Q_l = Q_0$ and $M_0 = M_l+lQ_l$. (Another way to obtain this familiar result is to require that q and m be controllable, i.e. functions of x only, and that they vanish in the particular problem at hand.)

The same reasoning can also be carried through in the dual case, for $\bar{c}$ and $\bar{g}$. However, if we do regard the original Lagrangian as the more important one, there arises a marked difference: whereas $\bar{q}$ is easily observable and thus can be required to vanish (L independent of x; $\bar{q}$ cannot be controlled otherwise), there appears to be no practical interpretation for the awkward condition $\bar{m} = 0$. The balance equation is, so far, with $(\,)^{\cdot} \equiv \mathrm{d}(\,)/\mathrm{d}w$,

$$[cQ+g(Qx+M)]'+\dot{\omega}'[\bar{c}\bar{Q}+\bar{g}(\bar{Q}w+\bar{M})]^{\cdot} \doteq -[cq+g(qx+m)]-\dot{\omega}'[\bar{c}\bar{q}+\bar{g}(\bar{q}w+\bar{m})],$$

and one sees immediately that it is a trivial identity, on the orbit. More interesting are the terms in k, $\bar{k}$, jointly. Even in K they multiply M, resp. $\bar{M}$, which are essentially uncontrollable. This could be settled by choosing $k+\bar{k} = 0$, but it appears to be more sensible to abandon duality and to consider only the original Lagrangian as essential. Thus we return to the original definition of K. To make progress, we must have a suitable explicit expression of L: we assume it is of the class

$$L_h = (1/h)s(x)(\overset{2}{w})^h-m(x)\overset{1}{w}-q(x)w.$$

With $\tau = \bar{k}x$ and

$$\delta\langle w\rangle = kw, \qquad \delta\langle \overset{1}{w}\rangle = (k-\bar{k})\overset{1}{w}, \qquad \delta\langle \overset{2}{w}\rangle = (k-2\bar{k})\overset{2}{w}$$

we find

$$K = [k-(2-1/h)\bar{k}]M\overset{2}{w}-km\overset{1}{w}-(k+\bar{k})qw-\bar{k}\overset{1}{w}\bar{q}x.$$

Now we get rid of M by choosing $k = (2-1/h)\bar{k}$, so that

$$K = -\bar{k}[(2-1/h)m\overset{1}{w}+(3-1/h)qw+\overset{1}{w}\bar{q}x].$$

This is observable, though it is controllable only by assuming $s = \text{const.}$ and by either considering the trivial Lagrangian $L = s(\overset{2}{w})^h/h$, or by considering $K = 0$ as a pair of ordinary differential equations to determine $q(x)$, $m(x)$, which appears to be rather artificial. Thus we rather leave K as it stands, and get the balance equation

$$[\bar{Q}x + (2-1/h)Q\mathring{\omega} + (1-1/h)M\mathring{\omega}']' \doteq -(2-1/h)m\mathring{\omega}' - (3-1/h)q\mathring{\omega} - \mathring{\omega}'\bar{q}x.$$

At this point, let us try the usual approach: given this Lagrangian L_h, we look for suitable τ, ξ so as to generalize the above balance equation. However, we still assume that τ, ξ depend only on x and w. We find

$$K = M[D^{(1)}D^{(0)}\xi - (2-1/h)(D^{(0)}\tau)\overset{2}{w} - (D^{(1)}D^{(0)}\tau)\overset{1}{w}] - \tau\overset{1}{w}\bar{q} - q(\xi + wD^{(0)}\tau) - mD^{(0)}\xi.$$

Expanding the total derivatives (and writing subscripts for the partial ones), the coefficient T of M becomes

$$T = \xi_{xx} + (\overset{1}{w})^2\xi_{ww} + 2\overset{1}{w}\xi_{xw} + \overset{2}{w}\xi_w - (2-1/h)\overset{2}{w}(\tau_x + \overset{1}{w}\tau_w) - \overset{1}{w}(\tau_{xx} + (\overset{1}{w})^2\tau_{ww} + 2\overset{1}{w}\tau_{xw} + \overset{2}{w}\tau_w).$$

This expression should vanish identically in $\overset{1}{w}$ and its powers, and in $\overset{2}{w}$ and the product $\overset{1}{w}\overset{2}{w}$: we must satisfy 6 equations separately. It is easily checked that the only solution is the one yielding the balance equation above, which thus has the significance of being unique for this Lagrangian. We stress here once more the importance of requiring $T \equiv 0$ on $\mathbb{E}^4$, so that if, in addition it so happens that $\bar{q} = q = m = 0$, we have $K \equiv 0$ and hence an evolution equation. Suppose, for instance, that we were to satisfy $T \doteq 0$ only on the orbit. This is easy : the Euler equation of the Lagrangian L_h is integrable by quadratures, so $\mathring{\omega}(x)$ is (in principle) determined. Thus we may as well assume that τ, ξ depend on x only; choose an arbitrary $\tau(x)$, insert it in the unexpanded form of T, and obtain $\xi(x)$ by two quadratures. Otherwise, insert an arbitrary $\tau(x, w)$ and the explicit derivatives of $\mathring{\omega}(x)$ in the explicit form of T, and solve the resulting p.d.e. for $\xi(x, w)$. Either way we have the balance equation

$$[\bar{Q}\tau + Q\xi + M(\xi' - \tau'\mathring{\omega}')]' \doteq -\tau\mathring{\omega}'\bar{q} - (\xi + \tau\mathring{\omega}')q - \xi'm,$$

in which the source term (r.h.s.) is observable and might vanish: but, we repeat, this equation is (in our opinion) trivial.

CONCLUSION

We have shown that duality, as understood here, can also be applied to higher-order Lagrangians. This suggests some interesting new interpretations. It is remarkable that the main expressions of transformation theory (Killing equation, Noetherian) turn out to be self-dual. However, it appears that duality is limited to general considerations (arbitrary Lagrangian). A complete theory of the dual body has yet to be found, as well as a clarification of the relation to the theory of forces on inhomogeneities.

REFERENCES

1. R. T. Shield, Inverse deformation results in finite elasticity. *Z.A.M.P.* **18**, 490–500 (1967).
2. P. Chadwick, Application of an energy-momentum tensor . . . *J. Elasticity* **5**, 249–258 (1975).
3. R. T. Shield, Conservation laws in finite elasticity. *Finite Elasticity*, AMD 27. A.S.M.E., New York (1977).
4. D. Rogula, Forces in material space. *Arch. Mech. Stosowanej* **29**, 705–713 (1977).
5. P. Casal, Interpretation of the Rice integral in continuum mechanics. *Lett. Appl. Engng Sci.* **16**, 335–347 (1978).
6. A. G. Herrmann, On physical and material conservation laws. *Proc. IUTAM Symp. Finite Elasticity*, Lehigh University (1980).
7. H. Zorski, Force on a defect in a non-linear elastic medium. *Int. J. Engng Sci.* **19**, 1573–1579 (1981).
8. A. G. Herrmann, On conservation laws of continuum mechanics. *Int. J. Solids Structures* **17**, 1–9 (1981).

9. A. G. Herrmann, Material momentum tensor and path-independent integrals. *Int. J. Solids Structures* **18**, 319–326 (1982).
10. R. Kienzler and G. Herrmann, A theory of beams with cracks. *Proc. AFMMS Conf.*, Freiburg/BRD, 20–24 June (1983).
11. A. G. Herrmann, On the Lagrangian formulation of continuum mechanics. *Physica* **118A**, 300–314 (1983).
12. J. D. Logan, *Invariant Variational Principles*. Academic Press, New York (1977).
13. W. Sarlet and F. Cantrijn, *Generalizations of Noether's Theorem* .... *SIAM Rev.* **23**, 467–494 (1981).
14. J. D. Eshelby, The elastic energy-momentum tensor. *J. Elasticity* **5**, 321–335 (1975).

APPENDIX

In order to facilitate the relation, we give a list of notation correspondences between this paper and the one by R. Kienzler, also in this issue.

Kienzler	Rösel
x, w	x, w
$\delta x = \Phi, \delta w = \psi$	$\tau, \delta w = \xi$
0	p
$\delta\psi = -\delta(w')$	$-\delta\langle \overset{1}{w} \rangle$
q, m, Q	q, m, Q
$-M$	M
$-V = qw - m\psi$	$V_c = qw + m\overset{1}{w}$
2	h
$W = (1/2)EI(x)(W'')^2$	$W_h = (1/h)s(x)(\overset{2}{w})^h$
b, B	$\overset{1}{w}\bar{q}, \bar{Q}$
$h = (5/2)qw + (3/2)mw'$	$(3-1/h)qw + (2-1/h)m\overset{1}{w}$
$-H = (3/2)Qw + (1/2)M\psi$	$(2-1/h)Qw + (1-1/h)M\overset{1}{w}$
$(W + V + 3Mw'')$	$\bar{m}$
$(M\psi^2)$	$\bar{M}$